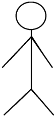
Historical Overview: the Complexity Theoretic Perspective on **PPAD** and Related Classes in **TFNP**

Mark Chen, Tianqi Yang

2024-04-11

Why This Presentation



Before This Talk

TFNP Expert

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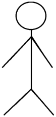

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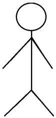

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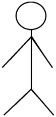

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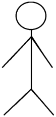

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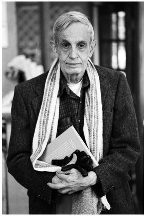
Goal

***TFNP** subclasses, like **PPAD**, are in the title of a lot of the papers we saw. But we have so far focused on stronger results (like showing hardness in **SVL** instead of **PPAD**), which were summarized in Yizhi's talk. Historical context and motivation are important, though.*

Line of Works



John Nash

Game Theory

Foundational Works

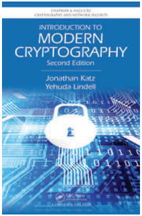
$\Longrightarrow$

C. Papadimitriou

Complexity

Theoretical Questions

$\Longrightarrow$



Entire Class

Cryptography

Specific Connections

Nash Equilibrium

Example, Concepts, and Existence

Example

	Coorporate	Defect
Coorporate	(2,2)	(3,0)
Defect	(0,3)	(1,1)

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	Go	Stop
Go	(-3,-3)	(0,1)
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Some key concepts & assumptions:

- Equilibrium := a set of stable strategies where no individual player has the incentive to change their strategy.
- Equilibrium strategies of other players are known to everyone.

Nash Equilibrium

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Theorem (Nash'51)

For every game, a **mixed Nash equilibrium**[a] *always exists (pure equilibrium, on the other hand, does not always exist; for example, rock-paper-scissors).*

[a] Mixed equilibrium is one where at least one player plays a randomized strategy.

Nash Equilibrium

As an algorithmic question

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Now that we have known the **existence** of Nash equilibrium since 70+ years ago, the key question left is:

Question

How **hard** *is it to compute the Nash equilibrium and how* **efficient** *can we make the computation?*

Not surprisingly, this gives us a **TFNP** problem (also efficiently verifiable, which is slightly harder to see for mixed strategy).

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Significance

*If we can indeed show that Nash equilibrium is **intractable**, Nash as a concept would be less useful as a ways to predict behaviors of players in the real world (since you can't do so efficiently; e.g., market prediction, etc.).*

Nash Equilibrium → Complexity Theoretic Question

Where (the heck) is it, then?

Remark

Again, by existence theorem, efficient verfiability and the search nature of the computation task (informally defined), **NASH** $\in$ **TFNP***.*

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Theorem (Theorem 2.1 [MP91])

Recall NO problem in **TFNP** *is* **NP***-complete, unless* **NP** $=$ **coNP**

So, without the latter condition, it is unlikely to show **NASH** is **NP**-complete (though searching for **NASH** equilibrium with natural additional properties (*e.g.*, maximized sum of utility) could be **NP**-complete).

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Remark

TFNP *is* **unlikely** *to contain any complete problems*[a].

[a] Semantic vs. syntactic. *e.g.*, **TFNP** & **NP** $\cap$ **coNP** are both semantic.

Complexity Theoretic Question

Where (the heck) is it, then?

Iron Chef's

"just basic chemistry"

Nash Equilibrium

Where (the heck) is it, then?

Papa's
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What if we, instead, find a subclass of **TFNP** problems (which is itself a subclass of **FNP**) based on the type of arguments used, where **NASH** is a complete problem[a]?

[a] Though they could have artificially defined a class of languages that reduce to **NASH** instead, defining based on types of totality arguments turned out to work very well as we have seen.

How It Started - Complexity Theoretic Question

Where (the heck) is **NASH**?

Theorem

[DGP09] As it turns out, **NASH** *is* **PPAD***-complete.*

*Note that, as a total problem, completeness must be shown through a two-way reduction, as all instances of total problems are guaranteed to be YES instances. Therefore, what we need to show is:

- **NASH** can be reduced to **EOTL**.
- **EOTL** can be reduced to **NASH**.

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Corollary

We have spent weeks constructing **SVL** *and* **rSVL** *hard instances from cryptographic assumptions, which easily translate to hardness in* **PPAD***. So, by completeness, all of such previous hardness results imply hardness in finding* **NASH** *equilibrium.*

Preliminaries of [DGP09]

We first define **NASH** and **Approximate-NASH** formally.

Definition ((Mixed) **NASH**)

Here are the set-ups:

- There are k players, and $p \in [k]$ denotes one of the players.
- Let S_p be a finite set of **strategies** that p can take, then $S = \prod_{p \in [k]} \mathbf{S}_p$ (Cartesian product). S is called **strategy profiles**.
- Let S_{-p} be the set of pure strategies of players other than p. Then, the **payoff** to p when p takes $s \in S_p$ and the other players take $s' \in S_{-p}$ is denoted by $u^p_{ss'} \geq 0$.

Now, let x^p_s denote the probability of p taking $s \in S_p$, finding **NASH** is the restraint problem:

$$x^p_s \geq 0 \text{ and } \sum_{s \in S_p} x^p_j = 1.$$

Frame Title

Definition ((Mixed) NASH, Continued)

Then, a k-mixed strategies is a **NASH** equilibrium if

$$\sum_{s \in S} u_s^p x_s \text{ is maximized } \forall p; x_s = \prod_{p \in [k]} x_{s_p}^p.$$

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Or, equivalently,

$$\sum_{s' \in S_{-p}} u_{ss'}^p x_{s'} > \sum_{s' \in S_{-p}} u_{s^\circ s'}^p x_{s'} \implies x_{s^\circ}^p = 0.$$

Preliminaries of [DGP09]

Definition (Approximate-NASH)

A set of mixed strategies x is an ϵ-**NASH** equilibrium if (with everything else the same):

$$\sum_{s' \in S_{-p}} u^p_{ss'} x_{s'} > \sum_{s' \in S_{-p}} u^p_{s^\circ s'} x_{s'} + \epsilon \implies x^p_{s^\circ} = 0.$$

Let's give some more intuition about this:

Remark

NASH *can be taken to mean requiring "no incentive to deviate," while* **Approximate-NASH** *is to require "low incentive to deviate." Say, if $\epsilon > 0$ is small, and then ϵ-***NASH** *equilibrium is a profile of mixed strategies where any player can improve its expected payoff by at most ϵ by switching to another strategy.*

Main Theorem

Theorem (Theorem 3.1 [DGP09])

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Recall the following idea, which is useful for reductions in both directions:

Theorem (Brouwer's Fixed Points Theorem)

Any continuous map from a compact and convex subset of the Euclidean space into itself always has a fixed point (one cannot map a circle continuously [rotate, flip, shrink and stretch] on itself without keeping some point fixed). A natural search problem is to find this point.

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Notice that for the problem to be tractable still, we need the point to be rational, and that is how we use the ϵ-approximate introduced for **NASH**.

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Definition (**BROUWER**(Π_F, K, ϵ))

Let Π_F be an efficient algorithm for the evaluation of $F : [0,1]^m \to [0,1]^m$. Let K be a constant so that F satisfies Lipschitz continuity:

$$\forall x_1, x_2 \in [0,1]^m : d(F(x_1), F(x_2)) \leq K \cdot d(x_1, x_2).$$

Let ϵ be the desired accuracy. Then, the search problem wants to output x such that $d(F(x), x) \leq \epsilon$.

Main Theorem Proof Overview

NASH $\longrightarrow$ **Approximate-NASH** $\longrightarrow$ **BROUWER** (**PPAD**-complete)

Main Theorem Direction 1 (Pre): **BROUWER** $\in$ **PPAD**

Theorem

BROUWER $\in$ **PPAD** *[Pap94]*.

- Triangulate the domain of F (fill up the domain with a mesh of tiny triangles and each triangle is a vertex of the graph).
- Color the vertices according to the direction in which F displaces them.
- Edges are defined with respect to the colors of the vertices.

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By a combinatorial argument called the **Sperner's lemma**, at least one triangle would satisfy this.

Main Theorem Direction 1: **NASH** $\in$ **PPAD**

Proof.

Suffices to show that **Approximate-NASH** $\leq$ **BROUWER**, which was first shown by Nash in 1950. Suppose the players in a game have chosen some mixed strategies. Unless the strategies are already at a Nash equilibrium, at least one of the players will be unsatisfied and will want to change to some other strategies.

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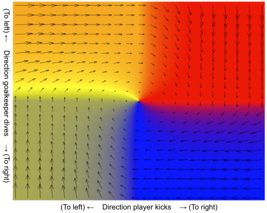
We see it as a "preference function" from the set of players' strategies to itself.



Magnitudes & directions are determined by $F_N(x) - x$, where $F_N(x)$ is Nash's function as a preference function for penalty shot game.

Clearly, **Approximate-NASH** equilibrium would be a ϵ-fixed point, and Brouwer's fixed point theorem guarantees its existence, **Approximate-NASH** $\equiv$ finding an approximate fixed point $\implies$ **NASH** $\in$ **PPAD**. ∎

Main Theorem Direction 2: **NASH** is **PPAD**-complete

Proof.

Similarly, we first show that **BROUWER** is **PPAD**-complete and then reduce **BROUWER** to **NASH**.

- (**BROUWER** is **PPAD**-complete): Need to show how to encode a **EOTL** graph as a continuous, easy-to-compute function F. This is non-trivial to show, but is given entirely in [DGP09].

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  - We let players be on every node on this data flow graph.
  - Thus, we simulate each arithmetic gate in the circuit by a game.
  - We compose the games to get the overall game.

  The specific ways to do so is non-trivial and are given entirely in [DGP09].

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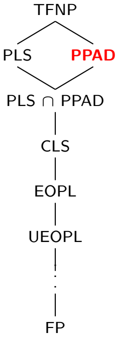
Note: Due to methods used, this only proves $k \geq 3$.

Summary of **PPAD**-complete problems we know now



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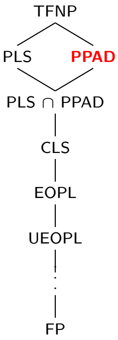
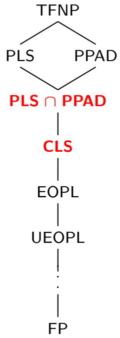


- **EOTL**.
- **NASH**.
- **BROUWER**.

Next, **CLS** $\in$ **PLS** $\cap$ **PPAD**

Now, we go deeper:



An Old Complete Problems in **PLS** $\cap$ **PPAD**

There was a reason to ask about optimizing for *continuous* functions, since the complete problems known back in the day [DP11] have, so to speak, an awkward flavor. Here's the general formula:

Example (**PPAD-OR-PLS**)

Given an instance $X \in$ **PPAD** and an instance $Y \in$ **PLS**.

- Either solve X.
- Or solve Y.

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Example (EITHER-FIXEDPOINT)

Let $\epsilon, \delta > 0$. Given three functions f, g and p. Here is the goal:

- Either find an approximate fixed point of f (or violation of f's δ-continuity).
- Or an approximate fixed point of g w.r.t. p (or violation of p's δ-continuity).

The reason why this appears awkward is because it is about finding fixed points of two unrelated functions.

CLS as a New Attempt; CLS $\subseteq$ PLS $\cap$ PPAD

So the question of interest at the time was: can we let f, g coincide in a single function, to make it more natural?

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Definition (CONTINUOUS-LOCALOPT)

Let $p : [2^n] \to \mathbb{R}$ and $f : [2^n] \to [2^n]$. Goal is to find $v \in [2^n]$ such that

- $p(f(v)) \geq p(v)$, or
- find a violation of continuity for f and/or p.

You can think of f as the successor function on a DAG line and p (potential) as the value of a node. Then, this condition is a search for the first node with minimal potential, or a sink.

CLS as a New Attempt; $\mathbf{CLS} \subseteq \mathbf{PLS} \cap \mathbf{PPAD}$

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You can think of f as the successor function on a DAG line and p (potential) as the value of a node. Then, this condition is a search for the first node with minimal potential, or a sink. So, this problem, which defines **CLS**, has a $\mathbf{PLS} \cap \mathbf{PPAD}$ flavor with an additional condition about continuity (not hard to show $\mathbf{CLS} \subseteq \mathbf{PLS} \cap \mathbf{PPAD}$).

Conjecture ([DP11], which is disproved by [FGHS22])

$\mathbf{CLS} \subsetneq \mathbf{PLS} \cap \mathbf{PPAD}$ *(because* **PLS** *and* **PPAD** *in general requires no continuity).*

CLS: **PPAD** $\cap$ **PLS** but with f, p Related to Each Other

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In this definition f and p can actually be related! One very natural such attempt is by using a Lipschitz continuous function f and its derivative $p = \lim_{\triangle x \to 0} \frac{f(x+\triangle x) - f(x)}{\triangle x}$. Here is how:

- If f is Lipschitz, then a fixed point must exist $|f(x) - x| = 0$, so it is in **PPAD**.
- Since f is Lipschitz continuous, its derivative must have the following property too: for some $\triangle x > 0$, $|f(x + \triangle x) - f(x)| < \epsilon$, which captures the definition for **PLS**.

CLS = PLS ∩ PPAD

Recall conjecture:

Conjecture ([DP11])

It was conjectured that **CLS** $\subsetneq$ **PLS** ∩ **PPAD**.

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It was conjectured that $\mathbf{CLS} \subsetneq \mathbf{PLS} \cap \mathbf{PPAD}$.

Next, we present the result that disproves this conjecture, i.e.

Theorem

$\mathbf{CLS} = \mathbf{PLS} \cap \mathbf{PPAD}$.

CLS = PLS $\cap$ PPAD: Preliminaries

In [FGHS22], it was shown that **GRADIENT-DESCENT**, which is a **CLS**-complete problem, is actually also (**PPAD** $\cap$ **PLS**)-complete.

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Definition (Gradient Descent)

Given a circuit for f and ∂f. Search for an extremum (minimum in particular) of a continuously differentiable function f over some domain D by starting at x_0 and iteratively as:

$$x_{k+1} \leftarrow x_k - \eta \nabla f(x_k).$$

Definition (Karush-Kuhn-Tucker (KKT) Optimality Condition)

Roughly, KKT optimality conditions assert that

- the gradient of f is 0 at x or
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Remark

Why does this resemble **PLS** *and* **PPAD** *problems?*

CLS $=$ **PLS** $\cap$ **PPAD**: Complete Search Problem

Definition (Set-Up)

Let $\epsilon, \eta > 0$, domain be D, $f \in C^1_L(D, \mathbb{R})$ (note ∇f is the gradient of f). The goal is to compute a point where gradient descent for f on D terminates.

CLS = PLS ∩ PPAD: Complete Search Problem

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Termination is determined using optimality conditions. In particular, if $x \in D$ and $x' = \prod_D (x - \eta \nabla f(x))$, then gradient descent should terminate if one of the following is found:

Definition (GD-LOCAL-SEARCH)

$f(x') \geq f(x) - \epsilon$. ϵ-approximate of the local minimum (**PLS**).

Definition (GD-FIXEDPOINT)

$|x - x'| - \epsilon$. ϵ-approximate of x-fixed point (**PPAD**).

Main Results of the Paper

Recall that GRADIENT-DESCENT as a search problem is defined by searching for either GD-LOCAL-SEARCH or GD-FIXEDPOINT. It turns out the following are true:

Theorem

Given a $f \in C^1_L(D, \mathbb{R})$ and its derivative as circuits, optimizing through GRADIENT-DESCENT is complete for **PLS** $\cap$ **PPAD**.

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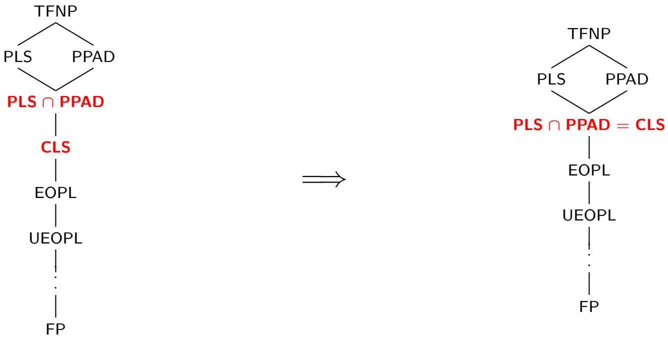
Corollary

CLS $=$ **PLS** $\cap$ **PPAD**. *So, with the weeks of results that we saw for constructing hard* **SVL** *and* **rSVL** *instances from various crypto assumptions, then all imply hardness in* **PLS** $\cap$ **PPAD**, *and so* **CLS** *and the GRADIENT-DESCENT problem.*

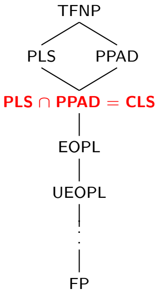
Thus, First Collapse



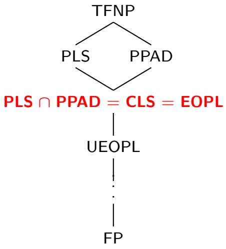
Now, We Have a Second Collapse [GHJ+22]

Theorem (Theorem 1 [GHJ+22])

EOPL $=$ **PLS** $\cap$ **PPAD**.



$\Longrightarrow$



End-of-Potential-Line (**EOPL**)

Recall what an **EOPL** is:

Definition (**EOPL**(S, P, x_0, p))

G is a DAG that is succinctly defined ($|V| = 2^n$) with in/out-degree of at most 1 [can be though of as a disjoint union of directed lines].

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Remark

EOPL $\subseteq$ **PPAD** $\cap$ **PLS** *should be intuitive, as it is equipped with* S, P *and* p *[***EOTL** *(***PPAD***-complete) is equipped with* S, P*, and SINK-OF-DAG (***PLS***-complete) is equipped with* p, S *and we can arbitrarily define* C *to never be violated].*

Theorem (**EOPL** = **CLS** = **PLS** ∩ **PPAD** [GHJ+22])

Proof.

Theorem 1 [GHJ+22]

$$\textbf{EOPL} \longrightarrow \textbf{CLS} = \textbf{PLS} \cap \textbf{PPAD}$$

∎

Theorem (**EOPL** = **CLS** = **PLS** ∩ **PPAD** [GHJ+22])

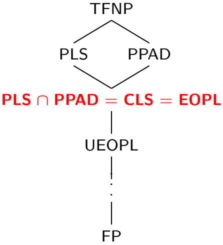
Proof.

Theorem 1 [GHJ+22]

EOPL ⟶ **CLS** = **PLS** ∩ **PPAD**

∎

So, the current state:



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